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The effective thermoelectroelastic properties of microinhomogeneous materials

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Abstract

The paper is concerned with composite materials which consist of a homogeneous matrix phase with a set of inclusions uniformly distributed in the matrix. The components of these materials are considered to be ideally elastic and exhibit piezoelectric properties. One of the variants of the self-consistent scheme, the Effective Field Method (EFM) is applied to calculate effective dielectric, piezoelectric and thermoelastic properties of such materials, taking into account the coupled electroelastic effects. At first the coupled thermoelectroelastic problem for a homogeneous medium with an isolated inclusion is solved. For an ellipsoidal inclusion and constant external field the solution of this problem is found in a closed analytic form. This solution is then used in the EFM to derive the effective thermoelectroelastic operator for the composite containing a random array of ellipsoidal inclusions. Explicit formulae for the electrothermoelastic constants are given for composites, reinforced by spheroidal inclusions. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Microinhomogeneous and composite materials made of two or more constituents find an increasingly wide application in modern technology. An important class of such materials is formed by the so-called matrix composites that comprise a homogeneous phase (matrix) containing an arrangement of filling particles of another component (inclusions).

As a rule, the microstructure of real composite materials is stochastic: random parameters characterize shapes, sizes and physical properties of inclusions. Also the distribution of the inclusions in the volume of the matrix is random. Hence the physical fields in these composites, even

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under deterministic external actions, are stochastic. The estimation of mean values (mathematical expectations) of these fields under the deterministic external actions is an important problem in the mechanics and physics of microinhomogeneous materials. From the solution of this homogenization problem one can find the effective properties of composite materials and one may substitute the so obtained homogeneous constitutive law in the macroscopic analysis of structures.

Theoretical approaches to the problem deal mostly with uncoupled elastic and electric theory (see, for example, the review articles by Willis, 1981; Kanaun and Levin, 1994). However, if the constituents exhibit electromechanical coupling, an external electric field can produce mechanical deformation which generate an internal stress field in the material and vice versa. Both of these fields can influence the macroscopic response of the material. Therefore, the coupling effects must be taken into account when estimating the overall properties of composites with piezoactive components.

The first theoretical treatment of this subject appears to be that of Newnham and co-workers (1978). Chan and Unsworth (1989) applied a Voigt type approximation to fibrous composites made of piezoelectric ceramic fibers and a polymer matrix, and they obtained simple explicit formulae. In a series of papers by Benveniste and Dvorak (1992), Benveniste (1993a, b, c, 1994a) exact relationships between the components of the tensors of effective material constants were derived (see also Dunn, 1993). Such relationships are especially useful if one is concerned with composites consisting of exactly two components. For fibrous composites Benveniste (1994b) was able to derive exact expressions for nine out of the ten effective constants by applying the composite cylinder assemblage model. Dunn and Taya (1993a) have generalized a number of popular micromechanical models, including the dilute concentration approach, the differential scheme and the Effective Medium Method. Particular attention has been given by those authors to the Mori–Tanaka Method (Dunn and Taya, 1993a; Dunn, 1993). The same approach was used in papers by Wang (1994) and Chen (1994).

In the mentioned articles one can find additional references to this field. Among them we cite only the monograph by Choroshun et al. (1989) where the method of conditional averaging has been used for predicting the overall electroelastic characteristics of piezoelectric composites. The application of variational principles and the Effective Medium Method for the estimation of electroelastic constants for composites and polycrystals can be found in the papers by Olson and Avellaneda (1992) and Dunn (1993, 1995).

All these approaches have certain limitations and drawbacks. So it is known, that the Effective Medium Method applied to matrix composites can lead to physically contradictory results, especially in the cases of large inclusion concentration and strong contrast in component properties. Concerning the Mori–Tanaka Method it is now clear (Kanaun and Levin, 1994; Ponte Castaneda and Willis, 1995) that it allows to consider only composites with aligned arrays of ellipsoidal inclusions.

In view of the situation, the present paper describes another self-consistent scheme, the Effective Field Method (EFM), which will be used for constructing the effective coupled electroelastic operator for piezoactive microinhomogeneous media. This method leads to physically reasonable results in the whole range of inclusions concentration. Furthermore it allows to consider in an averaged manner also the spatial correlation between the individual inclusions. The present article is based on an earlier paper (Levin, 1996) where the method was used for the derivation of electroelastic constants in isothermal problems. Here these results are expanded by taking into account the thermoeffects.

2. Equilibrium of a homogeneous electroelastic medium with a single inclusion

Let us consider a homogeneous elastic piezoelectric material. The linear constitutive relations of thermoelectroelasticity for this material have the form:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{ijk}E_k - \beta_{ij}\theta,$$
$$D_i = e^t_{ikl}\varepsilon_{kl} + \eta_{ik}E_k + \pi_i\theta \tag{2.1}$$

where σ and ε are the stress and strain tensors, **E** and **D** the electric field and electric displacement vectors, **C** is the tensor of elastic moduli at fixed electric field, η is the dielectric tensor (permittivity) at fixed strain, **e** is the tensor of piezoelectric constants characterizing coupled electroelastic effects (the transpose of **e**, written $\mathbf{e}^t$, is defined by $e^t_{ikl} = e_{kli}$), β is the tensor of temperature stress coefficients, π is the vector of pyroelectric constants, and θ denotes the temperature change from some reference temperature. Let us assume that mechanical and electrical fields do not affect the temperature distribution and that the temperature field, which is obtained by solving the uncoupled problem of heat-conduction, is known.

It is convenient to write the relations (2.1) in the following short form:

$$\boldsymbol{J} = \boldsymbol{L}\boldsymbol{F} + \boldsymbol{B}\theta,$$
$$\boldsymbol{J} = \begin{Vmatrix} \sigma \\ \mathbf{D} \end{Vmatrix}, \quad \boldsymbol{L} = \begin{Vmatrix} \mathbf{C} & -\mathbf{e} \\ \mathbf{e}^t & \eta \end{Vmatrix}, \quad \boldsymbol{F} = \begin{Vmatrix} \varepsilon \\ \mathbf{E} \end{Vmatrix}, \quad \boldsymbol{B} = \begin{Vmatrix} -\beta \\ \pi \end{Vmatrix}, \tag{2.2}$$

where the 'matrix' $\boldsymbol{L}$ and the 'vector' $\boldsymbol{B}$ have to be considered as linear operators, which transform the tensor–vector pair $[\sigma, \mathbf{D}]$ into another tensor–vector pair $[\varepsilon, \mathbf{E}]$. The reciprocal relations can be written as

$$\boldsymbol{F} = \boldsymbol{M}\boldsymbol{J} + \boldsymbol{R}\theta, \quad \boldsymbol{M} = \begin{Vmatrix} \mathbf{S} & \mathbf{g} \\ -\mathbf{g}^t & \kappa \end{Vmatrix}, \quad \boldsymbol{R} = \begin{Vmatrix} \rho \\ -\gamma \end{Vmatrix}, \tag{2.3}$$

where

$$\mathbf{S} = (\mathbf{C} + \mathbf{e}\eta^{-1}\mathbf{e}^t)^{-1}, \quad \kappa = (\eta + \mathbf{e}^t\mathbf{C}^{-1}\mathbf{e})^{-1}, \quad \mathbf{g} = \mathbf{S}\mathbf{e}\eta^{-1} = \mathbf{C}^{-1}\mathbf{e}\kappa,$$
$$\rho = \mathbf{S}(\beta - \mathbf{e}\eta^{-1}\pi), \quad \gamma = \kappa(\pi + \mathbf{e}^t\mathbf{C}^{-1}\beta). \tag{2.4}$$

Here **S** is the compliance at constant electric displacement, κ denotes the inverse of the permittivity tensor at constant mechanical stress, ρ stands for the tensor of thermal expansion coefficients, and γ is another form of the pyroelectric coefficients (at constant electric displacement).

Because the EFM is based on the solution of a one-particle problem, let us consider now an unbounded piezoactive medium with the thermoelectroelastic characteristics $\boldsymbol{L}^0$ and $\boldsymbol{B}^0$, containing the closed region V (inclusion) with different thermoelectroelastic properties $\boldsymbol{L}$ and $\boldsymbol{B}$. Such a problem has been considered by several authors (Deeg, 1980; Wang, 1992; Benveniste, 1992; Dunn and Taya 1993b). For completeness, however, we give a short derivation of the relevant results below.

We start with the following system of elastic and electric equilibrium equations of the coupled electroelastic theory for a heterogeneous medium:

$$\nabla[\boldsymbol{L}(x)\nabla\boldsymbol{f}(x)+\boldsymbol{B}(x)\theta(x)]=0,\quad \boldsymbol{f}(x)=\left\|\begin{matrix}u_i(x)\\-\varphi(x)\end{matrix}\right\|. \tag{2.5}$$

Here $u_i(x)$ is the vector of elastic displacements at a certain position x, $\varphi(x)$ is the electric potential and $\nabla_i=\partial/\partial x_i$.

For our present inclusion problem we can express the operators $\boldsymbol{L}(x)$ and $\boldsymbol{B}(x)$ in the form

$$\boldsymbol{L}(x)=\boldsymbol{L}^0+\boldsymbol{L}^1V(x),\quad \boldsymbol{B}(x)=\boldsymbol{B}^0+\boldsymbol{B}^1V(x),$$
$$\boldsymbol{L}^1=\boldsymbol{L}-\boldsymbol{L}^0,\quad \boldsymbol{B}^1=\boldsymbol{B}-\boldsymbol{B}^0, \tag{2.6}$$

where $V(x)$ is characteristic function of the region V. Then the problem of finding the solution $u_i(x)$ and $\varphi(x)$ can be reduced to a system of integral equations, which is equivalent to the initial system of differential eqns (2.5):

$$\boldsymbol{F}(x)=\boldsymbol{F}^0(x)+\int_V\boldsymbol{P}(x-x')[\boldsymbol{L}^1\boldsymbol{F}(x')+\boldsymbol{B}^1\theta(x')]\,\mathrm{d}x',$$
$$\boldsymbol{P}(x)=\boldsymbol{D}\boldsymbol{G}(x)\boldsymbol{D},\quad \boldsymbol{D}=\left\|\begin{matrix}\mathrm{def} & 0\\ 0 & \mathrm{grad}\end{matrix}\right\|. \tag{2.7}$$

Here $\boldsymbol{F}^0(x)$ is the vector of external elastic and electric fields, which would be in the medium without the inclusion. These fields satisfy the equations

$$\nabla\boldsymbol{L}^0\boldsymbol{F}^0(x)=-\nabla\boldsymbol{B}^0\theta(x) \tag{2.8}$$

and given conditions at infinity. The operator $\boldsymbol{G}(x)$ in (2.7) comprises the Green's functions of the coupled electroelastic differential operator, which satisfies the equation

$$\nabla\boldsymbol{L}^0\nabla\boldsymbol{G}(x)=\bar{\boldsymbol{I}}\delta(x),\quad \bar{\boldsymbol{I}}=\left\|\begin{matrix}-\delta_{ij} & 0\\ 0 & 1\end{matrix}\right\|, \tag{2.9}$$

where $\delta(x)$ is Dirac's function. For arbitrary anisotropy of the medium the operator $\boldsymbol{G}(x)$ is given by the expressions

$$\boldsymbol{G}(x)=\frac{1}{8\pi^2}\int_{|\xi|=1}\boldsymbol{G}(\xi)\delta(\xi\cdot x)\,\mathrm{d}S_\xi,\quad \boldsymbol{G}(\xi)=\left\|\begin{matrix}G_{ij}(\xi) & \gamma_i(\xi)\\ -\gamma_j^t(\xi) & g(\xi)\end{matrix}\right\|,$$
$$G_{ij}=\left(\Lambda_{ij}-\frac{1}{\lambda}H_ih_j\right)^{-1},\quad \gamma_j=\frac{1}{\lambda}h_iG_{ij},\quad g=-(\lambda+h_i\Lambda_{ij}^{-1}H_j)^{-1},$$
$$\Lambda_{ij}(\xi)=C^0_{ikjl}\xi_k\xi_l,\quad H_i(\xi)=e^0_{ikl}\xi_k\xi_l,\quad h_j(\xi)=e^{0t}_{ljk}\xi_k\xi_l,\quad \lambda(\xi)=\eta^0_{ij}\xi_i\xi_j. \tag{2.10}$$

For $x\in V$, the system of eqns (2.7) yield the fields $\varepsilon(x)$ and $E(x)$ inside the inclusion, thereupon the fields outside of V are determined uniquely.

Let now the inclusion have an ellipsoidal shape with semi-axes a_1, a_2, a_3. Then the domain V is defined by the relations $x_i(a^{-2})_{ij}x_j\leqslant 1$, $a_{ij}=a_i\delta_{ij}$ (no summing with respect to i!). In this case it can be shown that an integral operator with the kernel $\boldsymbol{P}(x)$ has the property of polynomial conservativity (Kunin and Sosnina, 1971; Asaro and Barnett, 1975). In particular, let the external fields and the temperature be homogeneous in the domain V ($F^0 = \text{const}$, $\theta = \text{const}$) and let this domain have a form of a sphere with radius a centered at the origin. If $F = \text{const}$ in V, the problem reduces to the evaluation of the integral (Willis, 1981)

$$\int_V \boldsymbol{P}(x-x')\,\mathrm{d}x' = \frac{1}{8\pi^2}\int_{|\xi|=1} \boldsymbol{P}(\xi)\,\mathrm{d}S_\xi \cdot \frac{\partial^2}{\partial p^2}\int_V \delta(p-\xi\cdot x')\,\mathrm{d}x',$$
$$p = \xi\cdot x, \quad \boldsymbol{P}(\xi) = \xi\boldsymbol{G}(\xi)\xi. \tag{2.11}$$

The integral over the domain V in (2.11) is equal to the area of intersection of the plane $\xi\cdot x = p$ with the sphere V; that is $\pi^2(a^2-p^2)$ if $|p| \leqslant a$ and zero if $p > a$. For $x \in V$, the second derivative of this integral equals -2π and the right-hand side of (2.11) is a constant. Similar results can be obtained for ellipsoids, which by means of a coordinate transformation $t_i = a_{ij}^{-1}x_j$ can be transformed to the unit sphere. In this case

$$\int_V \boldsymbol{P}(x-x')\,\mathrm{d}x' = -\boldsymbol{P}^0 = \text{const},$$
$$\boldsymbol{P}^0 = \frac{|\det a|}{4\pi}\int_{|\xi|=1} \boldsymbol{P}(\xi)\frac{\mathrm{d}S_\xi}{\rho^3(\xi)}, \quad \rho(\xi) = \sqrt{\xi_i(a^2)_{ij}\xi_j}. \tag{2.12}$$

Thus, for an external field $\boldsymbol{F}^0(x)$, and temperature field $\theta(x)$ which are homogeneous in V, the integral eqn (2.7) is transformed into an algebraic equation

$$\boldsymbol{F} = \boldsymbol{F}^0 - \boldsymbol{P}^0(\boldsymbol{L}^1\boldsymbol{F}+\boldsymbol{B}^1\theta). \tag{2.13}$$

Resolving this equation for $\boldsymbol{F}$, we can express the strain field ε and the electric field $\boldsymbol{E}$ via external fields ε^0, $\boldsymbol{E}^0$ and temperature θ

$$\boldsymbol{F} = \boldsymbol{A}(\boldsymbol{F}^0 - \boldsymbol{P}^0\boldsymbol{B}^1\theta), \quad \boldsymbol{A} = (\boldsymbol{I}+\boldsymbol{P}^0\boldsymbol{L}^1)^{-1},$$
$$\boldsymbol{I} = \left\| \begin{matrix} I_{ijkl} & 0 \\ 0 & \delta_{ik} \end{matrix} \right\|, \quad I_{ijkl} = \delta_{i(k}\delta_{l)j}. \tag{2.14}$$

We should mention that, for any two-phase composite, there is an exact relationship between the thermal part and the external loading part of (2.14) (see Benveniste, 1993b).[1]

3. Effective thermoelectroelastic properties of a medium with a random set of inclusions

We examine an unbounded elastic piezoelectric medium with properties $\boldsymbol{L}^0$ and $\boldsymbol{B}^0$, containing a spatially homogeneous random set of ellipsoidal inclusions which occupy a system of isolated regions V_k with characteristic functions $V_k(x)$, $k = 1, 2, \ldots$. The system of equations for the strain

[1] The authors are grateful to an anonymous referee who pointed out the equivalence of our expression with that exact result.

field $\varepsilon(x)$ and the electric field $\boldsymbol{E}(x)$ in the medium with inhomogeneities can be written in a form similar to (2.7)

$$\boldsymbol{F}(x) = \boldsymbol{F}^0(x) + \int_V \boldsymbol{P}(x-x')[\boldsymbol{L}^1(x')\boldsymbol{F}(x') + \boldsymbol{B}^1(x')\theta(x')]V(x')\,\mathrm{d}x'. \tag{3.1}$$

Here $V(x)$ denotes the characteristic function of the region $V = \Sigma_k V_k$, occupied by inclusions, $\boldsymbol{L}^1(x)$ and $\boldsymbol{B}^1(x)$ are functions, which coincide with the spatially constant values $\boldsymbol{L}^1 = \boldsymbol{L}^1(\omega_k)$ and $\boldsymbol{B}^1 = \boldsymbol{B}^1(\omega_k)$ when $x \in V_k$ (ω_k is a set of geometric parameters which characterize the orientation of principle anisotropy axes of k-th ellipsoid).

To solve the homogenization problem and to develop the macroscopic system of coupled electroelastic equations on the basis of eqn (3.1), we shall use the Effective Field approach (Kanaun and Levin, 1994). In this method every inclusion is considered as an isolated region in an otherwise homogeneous medium—the matrix of the composite. The presence of surrounding inclusions is taken into account by introducing an effective external field acting on this inclusion. In distinction to the traditional form of the EFM (Walpole, 1969; Levin, 1976; Markov, 1981) the effective field is considered to be random and a special technique is used for calculating its statistical moments.

Let us fix one of the typical realizations of a random set of inhomogeneities and consider an arbitrary k-th inclusion, occupying the region V_k. We denote the local external field acting on this inclusion by $\boldsymbol{F}^*_{(k)}(x)$. This field is defined in V_k and is composed of the external field $\boldsymbol{F}^0(x)$ and disturbances of the fields due to surrounding inclusions.

Let now $\boldsymbol{F}^*(x)$ be the field which coincides with the $\boldsymbol{F}^*_{(k)}(x)$ when $x \in V_k$. By the help of the definition

$$V(x,x') = \sum_{i \neq k} V_i(x'), \quad x \in V_k, \tag{3.2}$$

we may write for an arbitrary point x inside the domain V

$$\boldsymbol{F}^*(x) = \boldsymbol{F}^0(x) + \int \boldsymbol{P}(x-x')[\boldsymbol{L}^1(x')\boldsymbol{F}(x') + \boldsymbol{B}^1(x')\theta(x')]V(x,x')\,\mathrm{d}x'. \tag{3.3}$$

We suppose that the field $\boldsymbol{F}^*(x)$ has the same structure in any region occupied by the inclusions (this is the first hypotheses of the EFM). In particular, if this field is constant inside each region V_k (but may vary randomly from one inclusion to another) the connection between the field $\boldsymbol{F}(x) (x \in V)$ and $\boldsymbol{F}^*(x)$ is given by the relation (2.14), which has been obtained above by solving the one-particle problem for an ellipsoidal inhomogeneity

$$\boldsymbol{F}(x) = \boldsymbol{A}(x)[\boldsymbol{F}^*(x) - \boldsymbol{P}^0(x)\boldsymbol{B}^1(x)\theta(x)]. \tag{3.4}$$

Here $\boldsymbol{A}(x)$ and $\boldsymbol{P}(x)$ are functions which for $x \in V_k$ coincide with the constant operators $\boldsymbol{A} = \boldsymbol{A}(a_k, \omega_k)$ and $\boldsymbol{P}^0 = \boldsymbol{P}(a_k, \omega_k)$ defined in eqns (2.12) and (2.14). We note that the field $\boldsymbol{F}^*_{(k)}(x)$, being constant in ellipsoid V_k, may depend in general on the orientation ω_k of this region. Substitution of the expression (3.4) into the right-hand side of eqns (3.1) and (3.3) allows us to express the electroelastic fields at an arbitrary point of the medium by the local external field and temperature

$$F(x) = F^0(x) + \int P(x-x')[L^A(x')F^*(x') + B^A(x')\theta(x')]V(x')\,dx',$$

$$L^A(x) = L^1(x)A(x), \quad B^A(x) = \tilde{A}(x)B^1(x) \tag{3.5}$$

where

$$A = (I + P^0L^1)^{-1}, \quad \tilde{A} = (I + L^1P^0)^{-1}.$$

Furthermore we obtain the self-consistent equation for the field $F^*(x)$

$$F^*(x) = F^0(x) + \int P(x-x')[L^A(x')F^*(x') + B^A(x')\theta(x')]V(x,x')\,dx'. \tag{3.6}$$

When we are concerned with a random set of inclusions, $F(x)$ and $F^*(x)$ become random functions. By taking the ensemble average of both sides of eqn (3.5) we find

$$\langle F(x)\rangle = F^0(x) + \int P(x-x')\langle [L^A(x')F^*(x') + B^A(x')\theta(x')]V(x')\,|\,x'\rangle\,dx'. \tag{3.7}$$

Here the symbol $\langle \cdot\,|\,x'\rangle$ depicts the ensemble mean under the condition that x' is located in the region V, occupied by inclusions.

Let us now suppose that the random field $F^*(x)$ in the points of the region V_i is statistically independent of the physical properties of this region (the second main hypotheses of the EFM). In addition let the temperature field $\theta(x)$ be deterministic. This allows us to express the mean on the right-hand side of (3.7) as

$$\langle [L^A(x)F^*(x) + B^A(x)\theta(x)]V(x)\,|\,x\rangle = \langle L^A(x)V(x)\hat{F}^*(x,\omega)\rangle + \langle B^A(x)V(x)\rangle\theta(x). \tag{3.8}$$

Here we have defined: $\hat{F}^*(x,\omega) = \langle F^*(x)\,|\,x,\omega\rangle$, where the symbol $\langle\cdot\,|\,x,\omega\rangle$ depicts the mean under the condition that the point x is located in an inclusion with orientation ω. The mean $\hat{F}^*(x,\omega)$ will be called the effective field acting on the inclusion with orientation ω. For a spatially homogeneous set of inclusions, $B^A(x)$ is a homogeneous random function exhibiting the ergodic property. Using this property we obtain

$$\langle B^A(x)V(x)\rangle = \bar{B}^A, \quad \bar{B}^A = n_0\langle vB^A(a,\omega)\rangle. \tag{3.9}$$

Here n_0 is the numerical concentrations of inclusions, v is the volume of the typical inclusion, and the averaging of the right parts of the last expression goes over the random sizes, orientations and properties of the ellipsoidal inhomogeneities.

Taking into account (3.8) and (3.9), eqn (3.7) now has the form

$$\langle F(x)\rangle = F^0(x) + \int P(x-x')[T^*(x') + \bar{B}^A\theta(x')]\,dx'.$$

$$T^*(x') = \langle L^A(x')V(x')\hat{F}^*(x',\omega)\rangle. \tag{3.10}$$

It follows from this that the average field $\langle F(x)\rangle$ at an arbitrary point x of a composite material can be expressed by the moment of effective field $T^*(x')$. Equation (3.6) gives the possibility to determine this moment. After averaging both parts of (3.6) under the condition $x \in V(\omega)$, we can write

$$\hat{F}^*(x,\omega) = F^0(x) + \int P(x-x')\langle [L^A(x')F^*(x') + B^A(x')\theta(x')]V(x,x')\,|\,x';x,\omega\rangle\,dx'. \tag{3.11}$$

Here the symbol $\langle\cdot\,|\,x';x,\omega\rangle$ denotes the operation of averaging under the condition $x \in V(\omega)$, $x' \in V$. In general the mean $\langle\cdot\,|\,x';x,\omega\rangle$ differs from $\langle\cdot\,|\,x,\omega\rangle$ and eqn (3.1) turns out to be statistically unclosed. To close it we must invoke certain additional assumptions concerning the statistical properties of the field $\boldsymbol{F}^*(x)$. The simplest assumption is represented by analogue of the so-called 'quasicrystalline approximation' proposed by Lax (1951), according to which the means $\langle \cdot \mid x'; x, \omega \rangle$ and $\langle \cdot \mid x, \omega \rangle$ coincide. This results in

$$\hat{\boldsymbol{F}}^*(x,\omega) = \boldsymbol{F}^0(x) + \int \boldsymbol{P}(x-x')\langle [\boldsymbol{L}^A(x')\boldsymbol{F}^*(x') + \boldsymbol{B}^A(x')\theta(x')]V(x,x') \mid x,\omega\rangle \, dx'. \tag{3.12}$$

Assuming that the inclusion properties are statistically independent of their location in space, we may write the means on the right-hand side of (3.12) in the form

$$\langle [\boldsymbol{L}^A(x')\boldsymbol{F}^*(x') + \boldsymbol{B}^A(x')\theta(x')]V(x,x') \mid x,\omega\rangle = [\boldsymbol{T}^*(x') + \bar{\boldsymbol{B}}^A\theta(x')]\Psi_\omega(x,x')$$
$$\Psi_\omega(x,x') = \langle V(x)\rangle^{-1}\langle V(x,x') \mid x,\omega\rangle. \tag{3.13}$$

For a spatially homogeneous set of inclusions, the function $\Psi_\omega(x,x')$ depends only on the difference of arguments ($\Psi_\omega(x,x') = \Psi_\omega(x-x')$). This function characterizes the density of inhomogeneity distribution surrounding the typical inclusion of orientation ω. It defines the shape of the 'correlation hole', inside which a typical inclusion of orientation ω is located.

Equation (3.11) takes the form

$$\hat{\boldsymbol{F}}^*(x,\omega) = \boldsymbol{F}^0(x) + \int \boldsymbol{P}(x-x')[\boldsymbol{T}^*(x') + \bar{\boldsymbol{B}}^A\theta(x')]\Psi_\omega(x-x') \, dx'. \tag{3.14}$$

Eliminating the external field $\boldsymbol{F}^0(x)$ from eqns (3.10) and (3.14) we get the equation which couples the effective field $\hat{\boldsymbol{F}}^*(x,\omega)$ and the average field $\langle \boldsymbol{F}(x)\rangle$ in the composite

$$\hat{\boldsymbol{F}}^*(x,\omega) = \langle \boldsymbol{F}(x)\rangle - \int \boldsymbol{P}(x-x')[\boldsymbol{T}^*(x') + \bar{\boldsymbol{B}}^A\theta(x')]\Phi_\omega(x-x') \, dx',$$
$$\Phi_\omega(x) = 1 - \Psi_\omega(x). \tag{3.15}$$

Let us assume that there exists a linear transformation of the x-space that rearranges the function $\Phi_\omega(x)$ into a spherically symmetric one

$$y = b(\omega), \quad \Phi_\omega(b^{-1}(\omega)y) = \Phi_\omega(|y|). \tag{3.16}$$

In this case the ellipsoid defined by the equation $|b(\omega)x| = 1$ with semi-axes b_1, b_2, b_3 describes the form of the correlation hole.

For a spatially homogeneous random set of inclusions, $\Phi_\omega(x)$ is a smooth function, which quickly goes to zero outside a region having a size of the order of the correlation hole size. If we neglect the change of the fields $\hat{\boldsymbol{F}}^*(x,\omega)$ and $\theta(x)$ in this region, eqn (3.15) takes the form

$$\hat{\boldsymbol{F}}^*(x,\omega) = \langle \boldsymbol{F}(x)\rangle + \boldsymbol{P}^\Phi_\omega[\boldsymbol{T}^*(x) + \bar{\boldsymbol{B}}^A\theta(x)],$$
$$\boldsymbol{P}^\Phi_\omega = -\int \boldsymbol{P}(x)\Phi_\omega(x) \, dx. \tag{3.17}$$

Let us multiply both parts of eqn (3.17) by the operator $\boldsymbol{L}^A(x)V(x)$ and average the result over the ensemble of random orientation of inclusions. This can be written as

$$\boldsymbol{T}^*(x) = \bar{\boldsymbol{L}}^A\langle \boldsymbol{F}(x)\rangle + \langle \boldsymbol{L}^A(x)V(x)\boldsymbol{P}^\Phi_\omega(x)\rangle[\boldsymbol{T}^*(x) + \bar{\boldsymbol{B}}^A\theta(x)],$$
$$\bar{\boldsymbol{L}}^A = n_0\langle v\boldsymbol{L}^A(a,\omega)\rangle \tag{3.18}$$

where $\boldsymbol{P}^\Phi_\omega(x)$ is the function which coincides with $\boldsymbol{P}^\Phi_\omega$ inside the inclusion of orientation ω. Resolving this equation for $\boldsymbol{T}^*(x)$ we find

$$\boldsymbol{T}^*(x) = \boldsymbol{D}[\bar{\boldsymbol{L}}^A\langle \boldsymbol{F}(x)\rangle + \langle \boldsymbol{L}^A(x)V(x)\boldsymbol{P}^{\Phi}_{\omega}(x)\rangle \bar{\boldsymbol{B}}^A\theta(x)],$$
$$\boldsymbol{D} = (\boldsymbol{I} - \langle \boldsymbol{L}^A(x)V(x)\boldsymbol{P}^{\Phi}_{\omega}\rangle)^{-1}. \tag{3.19}$$

Substitution of this expression into the right-hand side of (3.10) gives

$$\langle \boldsymbol{F}(x)\rangle = \boldsymbol{F}^0(x) + \int \boldsymbol{P}(x-x')\boldsymbol{D}[\bar{\boldsymbol{L}}^A\langle \boldsymbol{F}(x')\rangle + \bar{\boldsymbol{B}}^A\theta(x')]\,\mathrm{d}x'. \tag{3.20}$$

Let us apply the operator $\nabla \boldsymbol{L}^0$ to both sides of the expression. Taking into account the relations (2.8) and (2.9), we find that the average elastic and electric fields in the composite material satisfy the equation

$$\nabla[\boldsymbol{L}^*\nabla\langle \boldsymbol{f}(x)\rangle + \boldsymbol{B}^*\theta(x)] = 0, \quad \boldsymbol{L}^* = \boldsymbol{L}^0 + \boldsymbol{D}\bar{\boldsymbol{L}}^A, \quad \boldsymbol{B}^* = \boldsymbol{B}^0 + \boldsymbol{D}\bar{\boldsymbol{B}}^A, \tag{3.21}$$

which coincides in form with the equilibrium equation of the coupled thermoelectroelastic theory for some homogeneous medium. The response of this medium to an external action is macroscopically identical with the reaction of the microinhomogeneous material. The quantities $\boldsymbol{L}^*$ and $\boldsymbol{B}^*$ in (3.21) represent the operators of effective thermoelastic characteristics of the piezoelectric composite.

4. Evaluation for transversely isotropic inclusions with spheroidal shape in an isotropic matrix

Now we are going to evaluate the general theory for inclusions which can be described as spheroids with semi-axes $a_1 = a_2 = a$ and a_3 corresponding to a rectangular coordinate system (x_1, x_2, x_3). Furthermore we assume that the matrix has isotropic properties with Lamé's constants λ_0, μ_0 and dielectric coefficient η_0. For this case the operator $\boldsymbol{P}$ of (2.12) is given in the Appendix.

The operators $\boldsymbol{A}$ and $\boldsymbol{L}^A$ can be represented in the following compact form

$$\boldsymbol{A} = \begin{Vmatrix} \bar{\mathbf{A}} & \mathbf{H} \\ -\mathbf{h}^t & \bar{\alpha} \end{Vmatrix}, \quad \boldsymbol{L}^A = \begin{Vmatrix} \mathbf{C}^A & -\mathbf{e}^A \\ \mathbf{e}^{At} & \eta^A \end{Vmatrix} \tag{4.1}$$

where we have used

$$\bar{\mathbf{A}} = (\mathbf{I} + \mathbf{P}\bar{\mathbf{C}})^{-1}, \quad \bar{\alpha} = (\bar{\mathbf{I}} + \mathbf{p}\bar{\eta})^{-1}, \quad \mathbf{H} = \mathbf{A}\mathbf{P}\mathbf{e}\bar{\alpha}, \quad \mathbf{h} = \alpha\mathbf{p}\mathbf{e}^t\bar{\mathbf{A}}$$
$$\bar{\eta} = \eta^1 + \mathbf{e}^t\mathbf{A}\mathbf{P}\mathbf{e}, \quad \bar{\mathbf{C}} = \mathbf{C}^1 + \mathbf{e}\alpha\mathbf{p}\mathbf{e}^t, \quad \mathbf{A} = (\mathbf{I} + \mathbf{P}\mathbf{C}^1)^{-1}, \quad \alpha = (\bar{\mathbf{I}} + \mathbf{p}\eta^1)^{-1}, \quad \bar{\mathbf{I}} = \delta_{ij}$$
$$\mathbf{C}^A = \bar{\mathbf{C}}\bar{\mathbf{A}}, \quad \mathbf{e}^A = \mathbf{A}^t\mathbf{e}\bar{\alpha}, \quad \eta^A = \bar{\eta}\bar{\alpha}. \tag{4.2}$$

Let the inclusions have hexagonal symmetry of the class 6 mm with the symmetry axis of infinite order directed along x_3. Then the tensors $\mathbf{C}$, $\mathbf{e}$, η, β and π can be decomposed according to a special tensor basis [see (A.2) in the Appendix]

$$\mathbf{C} = k\mathbf{T}^2 + 2m(\mathbf{T}^1 - \tfrac{1}{2}\mathbf{T}^2) + l(\mathbf{T}^3 + \mathbf{T}^4) + 4\mu\mathbf{T}^5 + n\mathbf{T}^6,$$
$$\mathbf{e} = e_1\mathbf{U}^1 + e_2\mathbf{U}^2 + e_3\mathbf{U}^3, \quad \eta = \eta_1\mathbf{t}^1 + \eta_2\mathbf{t}^2, \quad \beta = \beta_1\mathbf{t}^1 + \beta_2\mathbf{t}^2, \quad \pi_i = \pi m_i. \tag{4.3}$$

Here k, m, l, μ, n are five independent elastic moduli of the transversely isotropic medium, e_1, e_2, e_3 are three piezoelectric constants, η_1, η_2 are two dielectric coefficients, β_1, β_2 are two temperature stress coefficients, and π is the pyroelectric constant.

The coefficients of the expansions in (4.3) can be expressed via the 'common' components of the tensors **C**, **e** and η in the following way

$$k = \tfrac{1}{2}(C_{11}+C_{12}),\quad m = \tfrac{1}{2}(C_{11}-C_{12}),\quad l = C_{13},\quad \mu = C_{44},\quad n = C_{33}$$
$$e_1 = e_{31},\quad e_2 = e_{15},\quad e_3 = e_{33},\quad \eta_1 = \eta_{33},\quad \eta_2 = \eta_{11} = \eta_{22}. \tag{4.5}$$

Completing the calculation provided by the formulae (4.1) and (4.2) we have

$$\bar{\mathbf{A}} = \bar{A}_1\mathbf{T}^2 + \bar{A}_2\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) + \bar{A}_3\mathbf{T}^3 + \bar{A}_4\mathbf{T}^4 + \bar{A}_5\mathbf{T}^5 + \bar{A}_6\mathbf{T}^6,$$
$$\bar{A}_1 = \frac{1}{2\Delta}(1+P_6\bar{n}+2P_3\bar{l}),\quad \bar{A}_2 = (1+2m_1P_2)^{-1},\quad \bar{A}_3 = -\frac{1}{\Delta}(2P_1\bar{l}+P_3\bar{n}),$$
$$\bar{A}_4 = -\frac{1}{\Delta}(2P_3\bar{k}+P_6\bar{l}),\quad \bar{A}_5 = 2(1+\bar{\mu}P_5)^{-1},\quad \bar{A}_6 = \frac{2}{\Delta}\left(\frac{1}{2}+2P_1\bar{k}+P_3\bar{l}\right)$$
$$\Delta = 2\left[\left(\frac{1}{2}+2P_1\bar{k}+P_3\bar{l}\right)(1+P_6\bar{n}+2P_3\bar{l}) - (2P_1\bar{l}+P_3\bar{n})(2P_3\bar{k}+P_6\bar{l})\right] \tag{4.6}$$

where

$$\bar{k} = k^1 + \alpha_1 p_1 (e_1)^2,\quad \bar{l} = l^1 + \alpha_1 p_1 e_1 e_3,\quad \bar{\mu} = \mu^1 + \alpha_2 p_2 (e_2)^2,$$
$$\bar{n} = n^1 + \alpha_1 p_1 (e_3)^2,\quad \alpha_1 = (1+p_1\eta_1^1)^{-1},\quad \alpha_2 = (1+p_2\eta_2^1)^{-1}$$
$$k^1 = k-\lambda_0-\mu_0,\quad m^1 = m-\mu_0,\quad l^1 = l-\lambda_0,\quad \mu^1 = \mu-\mu_0,$$
$$n^1 = n-\lambda_0-2\mu_0,\quad \eta_1^1 = \eta_1-\eta_0,\quad \eta_2^1 = \eta_2-\eta_0. \tag{4.7}$$

The quantities $P_1,\ldots,P_6, p_1$ and p_2 are given in Appendix (A.4).

The remaining tensors in the operator $\boldsymbol{A}$ in (4.1) have the form

$$\bar{\alpha} = \bar{\alpha}_1\mathbf{t}^1 + \bar{\alpha}_2\mathbf{t}_2$$
$$\bar{\alpha}_1 = (1+p_1\bar{\eta}_1)^{-1},\quad \bar{\alpha}_2 = (1+p_2\bar{\eta}_2)^{-1},$$
$$\bar{\eta}_1 = \eta_1^1 + 4(2P_1A_1+P_3A_3)(e_1)^2 + 2(2P_3A_1+2P_1A_4+P_6A_3+P_3A_6)e_1e_3$$
$$+(P_6A_6+2P_3A_4)(e_3)^2,\quad \bar{\eta}_2 = \eta_2^1 + \tfrac{1}{2}P_5A_5(e_2)^2$$
$$\mathbf{H} = H_1\mathbf{U}^1 + H_2\mathbf{U}^2 + H_3\mathbf{U}^3$$
$$H_1 = \bar{\alpha}_1[2(2P_1A_1+P_3A_3)e_1 + (2P_3A_1+P_6A_3)e_3],\quad H_2 = \tfrac{1}{4}\bar{\alpha}_2P_5A_5e_2,$$
$$H_3 = \bar{\alpha}_1[2(2P_1A_4+P_3A_6)e_1 + (P_6A_6+2P_3A_4)e_3],$$
$$\mathbf{h}^t = h_1\mathbf{U}^{1t} + h_2\mathbf{U}^{2t} + h_3\mathbf{U}^{3t},$$
$$h_1 = \alpha_1p_1(2\bar{A}_1e_1+\bar{A}_4e_3),\quad h_2 = \tfrac{1}{2}\alpha_2p_2\bar{A}_5e_2,\quad h_3 = \alpha_1p_1(2\bar{A}_3e_1+\bar{A}_6e_3) \tag{4.8}$$

where the components of the tensor A_{ijkl} (i.e. $A_1, A_2, \ldots, A_6$) are obtainable from the components of the tensor $\bar{A}_{ijkl}$ (4.6) by putting $e_1 = e_2 = e_3 = 0$.

Let us now represent the formulae for the components of tensors appearing in the operator $\boldsymbol{L}^A$ in (4.1)

$$\mathbf{C}^A = k_A\mathbf{T}^2 + 2m_A\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) + l_A(\mathbf{T}^3+\mathbf{T}^4) + 4\mu_A\mathbf{T}^5 + n_A\mathbf{T}^6$$

$$k_A = \frac{\bar{k}}{\Delta}\left[1+P_6\left(\bar{n}-\frac{\bar{l}^2}{\bar{k}}\right)\right], \quad m_A = m_1(1+2m_1P_2)^{-1},$$

$$\mu_A = \bar{\mu}(1+P_5\bar{\mu})^{-1}, \quad l_A = \frac{1}{\Delta}[\bar{l}-2P_3(\bar{k}\bar{n}-\bar{l}^2)],$$

$$n_A = \frac{1}{\Delta}[\bar{n}+4P_1(\bar{k}\bar{n}-\bar{l}^2)],$$

$$\mathbf{e}^A = e_1^A\mathbf{U}^1 + e_2^A\mathbf{U}^2 + e_3^A\mathbf{U}^3,$$

$$e_1^A = (2A_1e_1+A_4e_3)\bar{\alpha}_1, \quad e_2^A = \frac{1}{2}A_5\bar{\alpha}_2e_2, \quad e_3^A = (2A_3e_1+A_6e_3)\bar{\alpha}_1,$$

$$\eta^A = \eta_1^A\mathbf{t}^1 + \eta_2^A\mathbf{t}^2$$

$$\eta_1^A = \bar{\eta}_1(1+p_1\bar{\eta}_1)^{-1}, \quad \eta_2^A = \bar{\eta}_2(1+p_2\bar{\eta}_2)^{-1}. \tag{4.9}$$

Analogous calculations allow us to determine the 'vector' $\boldsymbol{B}^A$:

$$\boldsymbol{B}^A = \left\| \begin{matrix} -\beta_{ij}^A \\ \pi_i^A \end{matrix} \right\|,$$

$$\beta_{ij}^A = \beta_1^A t_{ij}^1 + \beta_2^A t_{ij}^2,$$

$$\beta_1^A = \beta_1^1\bar{A}_6 + 2\beta_2^1\bar{A}_3 - \pi h_3, \quad \beta_2^A = \beta_1^1\bar{A}_4 + 2\beta_2^1\bar{A}_1 - \pi h_1,$$

$$\pi_i^A = \pi_A m_i$$

$$\pi_A = \pi\bar{\alpha}_1 + \beta_1^1 H_3 + 2\beta_2^1 H_1$$

$$\beta_1^1 = \beta_1 - \beta_0, \quad \beta_2^1 = \beta_2 - \beta_0 \tag{4.10}$$

where β_0 is the thermal stress coefficient of the matrix.

For simplicity we assume that all inclusions are homogeneously and isotropically distributed in the matrix and have identical thermo-electroelastic properties, volume v, and aspect ratio a/a_3. Then the only averaging remains over the random orientations of inclusions on the right-hand side of the formulae (3.9) and (3.18).

Two limiting cases can be considered.

4.1. Random orientations of inclusions

The orientation of spheroidal transversely isotropic inclusions is determined unequivocally by the orientation of the vector m. If this orientation is random then the bases (A.2) are also random.

In the case of a homogeneous distribution of inclusions orientations (statistical isotropy) the averaged tensors are given by

$$\langle \mathbf{T}^1(m)\rangle = \tfrac{1}{45}(10\mathbf{E}^1+21\mathbf{E}^2),\quad \langle \mathbf{T}^2(m)\rangle = \tfrac{2}{45}(10\mathbf{E}^1+3\mathbf{E}^2)$$

$$\langle \mathbf{T}^3(m)\rangle = \langle \mathbf{T}^4(m)\rangle = \tfrac{2}{45}(5\mathbf{E}^1-3\mathbf{E}^2),\quad \langle \mathbf{T}^5(m)\rangle = \tfrac{1}{5}\mathbf{E}^2,$$

$$\langle \mathbf{T}^6(m)\rangle = \tfrac{1}{45}(5\mathbf{E}^1+6\mathbf{E}^2)$$

$$\langle t^1_{ij}(m)\rangle = \tfrac{1}{3}\delta_{ij},\quad \langle t^2_{ij}(m)\rangle = \tfrac{2}{3}\delta_{ij}$$

$$E^1_{ijkl} = \delta_{ij}\delta_{kl},\quad E^2_{ijkl} = I_{ijkl}-\tfrac{1}{3}E^1_{ijkl}. \tag{4.11}$$

We may choose the shape of the correlation hole as a spheroid with an aspect ratio a/a_3 coinciding with those of the inclusion. Then we have for the operator $\boldsymbol{P}^{\Phi}_{\omega}$ from (3.17)

$$\boldsymbol{P}^{\Phi}_{\omega} = \boldsymbol{P}^0. \tag{4.12}$$

By averaging the operator $\boldsymbol{L}^A(x)$ and the vector $\boldsymbol{B}^A(x)$ by the help of expressions (4.11), we obtain

$$\bar{\boldsymbol{L}}^A = c\left\|\begin{matrix}\bar{\mathbf{C}}^A & 0\\ 0 & \bar{\eta}^A\end{matrix}\right\|,\quad \bar{\boldsymbol{B}}^A = c\left\|\begin{matrix}-\bar{\beta}^A\\ 0\end{matrix}\right\| \tag{4.13}$$

where

$$\bar{\mathbf{C}}^A = \bar{K}_A\mathbf{E}^1+2\bar{\mu}_A\mathbf{E}^2,\quad \bar{\eta}^A_{ij} = \bar{\eta}_A\delta_{ij},\quad \bar{\beta}^A_{ij} = \bar{\beta}^A\delta_{ij},$$

$$\bar{K}_A = \tfrac{1}{9}(4k_A+4l_A+n_A),\quad \bar{\mu}_A = \tfrac{1}{15}(k_A+6m_A-2l_A+6\mu_A+n_A),$$

$$\bar{\eta}_A = \tfrac{1}{3}(\eta^A_1+2\eta^A_2),\quad \bar{\beta}_A = \tfrac{1}{3}(\beta^A_1+2\beta^A_2) \tag{4.14}$$

and $c = n_0 v$ is the volume concentration of inclusions.

The composite material is macroscopically isotropic and its two effective elastic moduli (volume K^* and shear μ^*), dielectric coefficient η^* and thermal stresses coefficient β^* are determined by the expressions

$$K^* = K_0+c\bar{K}_A(1-3cB_k)^{-1},\quad \mu^* = \mu_0+c\bar{\mu}_A(1-2cB_\mu)^{-1},$$

$$\eta^* = \eta_0+c\bar{\eta}_A(1-cb)^{-1},\quad \beta^* = \beta_0+c\bar{\beta}_A(1-3cB_k)^{-1} \tag{4.15}$$

with

$$B_k = \tfrac{1}{9}[4(2P_1+P_3)k_A+2(2P_1+3P_3+P_6)l_A+(2P_3+P_6)n_A],$$

$$B_\mu = \tfrac{1}{15}[2(P_1-P_3)k_A+6P_2m_A+(3P_3-2P_1-P_6)l_A+3P_5\mu_A+(P_6-P_3)n_A],$$

$$b = \tfrac{1}{3}(p_1\eta^A_1+2p_2\eta^A_2) \tag{4.16}$$

($K_0 = \lambda_0+2/3\mu_0$ is the bulk elastic modulus of the matrix).

4.2. Unidirectionally oriented spheroidal inclusions

We consider a second special case where the identical spheroidal inclusions have random positions but identical (parallel) orientation. In this case the general expressions for the effective electroelastic characteristics of the composite (3.21) can be slightly simplified

$$\boldsymbol{L}^* = \boldsymbol{L}^0 + c\boldsymbol{L}^1[\boldsymbol{I} + (1-c)\boldsymbol{P}^0\boldsymbol{L}^1]^{-1}$$

$$\boldsymbol{B}^* = \boldsymbol{B}^0 + c[\boldsymbol{I} + (1-c)\boldsymbol{L}^1\boldsymbol{P}^0]^{-1}\boldsymbol{B}^1. \quad (4.17)$$

In accordance with these formulae the composite material exhibits macroscopically transversely isotropic symmetry and its thermoelectroelastic constants are expressed as

$$\boldsymbol{L}^* = \begin{Vmatrix} \mathbf{C}^* & -\mathbf{e}^* \\ \mathbf{e}^{*t} & \eta^* \end{Vmatrix}, \quad \boldsymbol{B}^* = \begin{Vmatrix} -\beta^* \\ \pi^* \end{Vmatrix} \quad (4.18)$$

where

$$\mathbf{C}^* = k^*\mathbf{T}^2 + 2m^*(\mathbf{T}^1 - \tfrac{1}{2}\mathbf{T}^2) + l^*(\mathbf{T}^3 + \mathbf{T}^4) + 4\mu^*\mathbf{T}^5 + n^*\mathbf{T}^6,$$

$$\mathbf{e}^* = e_1^*\mathbf{U}^1 + e_2^*\mathbf{U}^2 + e_3^*\mathbf{U}^3, \quad \eta^* = \eta_1^*\mathbf{t}^1 + \eta_2^*\mathbf{t}^2,$$

$$\beta^* = \beta_1^*\mathbf{t}^1 + \beta_2^*\mathbf{t}^2, \quad \pi_i^* = \pi^* m_i$$

$$k^* = \lambda_0 + \mu_0 + ck_A(c), \quad m^* = \mu_0 + cm_A(c), \quad l^* = \lambda_0 + cl_A(c),$$

$$\mu^* = \mu_0 + c\mu_A(c), \quad n^* = \lambda_0 + 2\mu_0 + cn_A(c), \quad e_i^* = ce_i^A(c) \quad (i = 1, 2, 3),$$

$$\eta_1^* = \eta_0 + c\eta_1^A(c), \quad \eta_2^* = \eta_0 + c\eta_2^A(c),$$

$$\beta_1^* = \beta_0 + c\beta_1^A(c), \quad \beta_2^* = \beta_0 + c\beta_2^A(c), \quad \pi^* = c\pi_A(c). \quad (4.19)$$

In these equations the quantities k_A, m_A, l_A, μ_A, n_A, e_i^A, η_k^A, π_A are now functions of the volume fraction c. They can be obtained from the corresponding expressions in (4.9) and (4.10) by simply replacing the quantities $P_1, \ldots, P_6$ and p_1, p_2 by $(1-c)P_i$ $(i = 1, 2, \ldots, 6)$ and $(1-c)p_k$ $(k = 1, 2)$.

The above expressions coincide with the results obtained by the Mori–Tanaka Method. This method is based on the assumption that every inclusion behaves as an isolated one in the matrix of the composite and that all inclusions undergo a constant and identical external electroelastic field in the matrix. This field is assumed to coincide with the average field in the matrix. It can be shown that the Mori–Tanaka Method leads to results identical with those of the present EFM if we accept as additional assumption that the shape of the correlation hole coincides with the shape of a typical inclusion. In particular we can demonstrate that under this assumption and for continuous fibers (aspect ratio $\gamma \to 0$) the present results coincide with the analytical predictions by Chen (1994). In the general case the shapes of inclusions and the correlation hole can be different and the operators $\boldsymbol{P}$ and $\boldsymbol{P}^{\Phi}$ are not the same.

5. Numerical results and discussion

Subsequently we present some typical predictions of the theory by considering a polymer (epoxy) matrix which contains a variable volume fraction of piezoactive particles (Lead–Zirconium–

Table 1
Electroelastic material constants

	C_{11} GPa	C_{12} GPa	C_{13} GPa	C_{33} GPa	C_{44} GPa	e_{31} Cm^{-2}	e_{33} Cm^{-2}	e_{15} Cm^{-2}	η_{11} ε_0	η_{33} ε_0
Polymer	8.0	4.4	4.4	8.0	1.8	0	0	0	4.2	4.2
PZT	148	76.2	74.2	131	25.4	−2.1	9.5	9.2	460	235
BT	150	66.0	66.0	146	44.0	−4.3	17.5	11.4	1115	1260

Where $\varepsilon_0 = 8.85\ 10^{-12}$ As/V m is the permittivity of the free space.

Table 2
Thermal material constants

	ρ_{11} $10^{-6}\ K^{-1}$	ρ_{33} $10^{-6}\ K^{-1}$	γ_3 $10^{3}\ V\ m^{-1}\ K^{-1}$
Polymer	60	60	0
BT	8.53	1.99	−13.3

Titanate (PZT) or Barium–Titanate (BT)). The materials data are shown in Tables 1 and 2. For simplicity and for comparison purposes we restrict ourselves to a set of material data which has been used previously by other authors (Dunn and Taya, 1993a; Dunn, 1993).

The piezoactive inclusions are approximated by spheroids with different aspect ratios. The axis of rotational symmetry is denoted by x_3, and it coincides with the poling direction. The inclusions are either aligned with the inclusions three-axis parallel to the sample three-axis or they are randomly oriented such that the composite exhibits macroscopically isotropic properties. The resulting equations which have to be evaluated are then given by (4.18) and (4.15), respectively.

Figure 1 shows the effective axial permittivity of a composite with aligned inclusions. We find a strong dependence on the shape of the inclusions, whereas the transverse permittivity is only weakly dependent on shape (not displayed here, the corresponding curves are very near to the result for spherical particles in Fig. 1).

Similar results are obtained for the uniaxial tension modulus at constant electric field (Fig. 2). One observes, however, a somewhat surprising crossing of the predicted curves when the volume fraction $c \to 1$. Such a behavior neither occurs in the uncoupled case nor is it observed for $1/s_{33}$, which is the uniaxial modulus at constant electric displacement [cf (2.4)]. The reason for this behavior near to $c = 1$ lies in the special piezoelectric coupling properties: the resulting effective modulus is a complicated function of the uniaxial modulus of a pure PZT ceramic, which may vary between 139 or 82 GPa at constant electric displacement or constant electric field, respectively, and it depends on the value of the effective piezoelectric coupling constant. The latter shows a particular strong variation near $c = 1$ (cf Figs 3 and 4) especially for spherical particles, and this should be the reason for the non-monotone behavior of the uniaxial modulus.

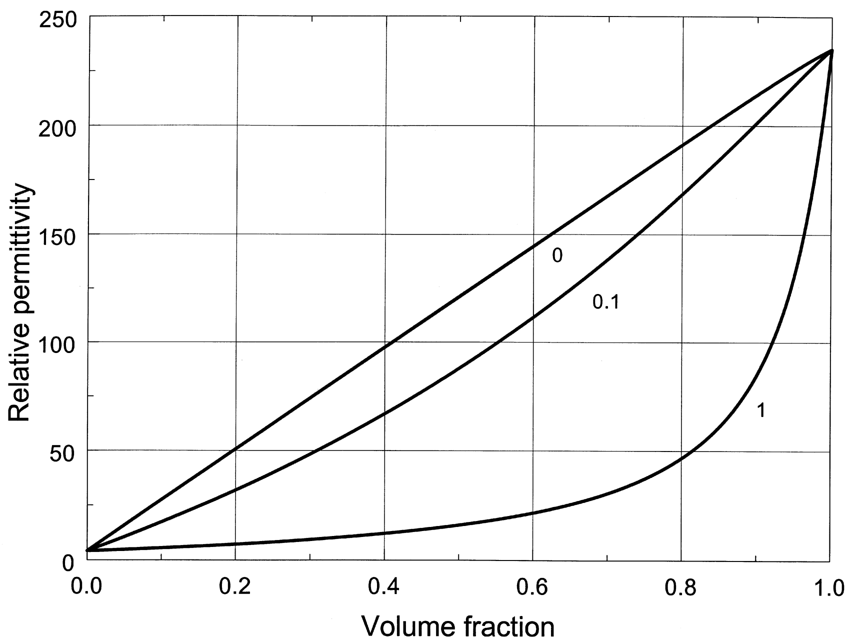

Fig. 1. Effective relative dielectric constant η_{33}/ε_0 at fixed strain of a PZT particle reinforced polymer (aligned orientation of inclusions, parameter = particle aspect ratio a/a_3).

Of course the limit $c \to 1$ looks artificial since we then are concerned with a nearly homogeneous material where the piezoelectric inclusions are still separated by a thin layer of matrix material. Nevertheless such a composite is realizable when an appropriate size distribution of inclusions is realized (the so-called composite sphere assemblage geometry, which can be applied also to ellipsoidal inclusions), and the theory predicts the right limit for $c = 1$.

The piezoelectric coupling constants are shown in Fig. 3. For an aspect ratio 0.1, the result can be compared with the theoretical prediction of Dunn and Taya (1993a) whose investigation was based on the Mori–Tanaka Method. According to the remark at the end of the preceding section, we obtain complete coincidence.

Besides e_{ik} it is also interesting to consider the alternative coupling constant d_{ik} which is defined by

$$\mathbf{d} = \mathbf{C}^{-1}\mathbf{e}. \tag{5.1}$$

In order to compare the predictions of the present model with experimental results, we adopt slightly different properties of the piezoelectric inclusions. Following Chan and Unsworth (1989) and Dunn and Taya (1993a) we put $e_{33} = 12.3$ C/m^2 instead of the value displayed in Table 1. Figure 4 shows the results. For continuous fibers we again get complete coincidence with the Mori–Tanaka results by Dunn and Taya (1993a) and there is also a relatively good accordance with the experimental results.

Next we turn to an isotropic composite. Although such a composite does not exhibit macroscopic piezoelectric properties it is interesting to investigate how much the effective permittivity is influ-

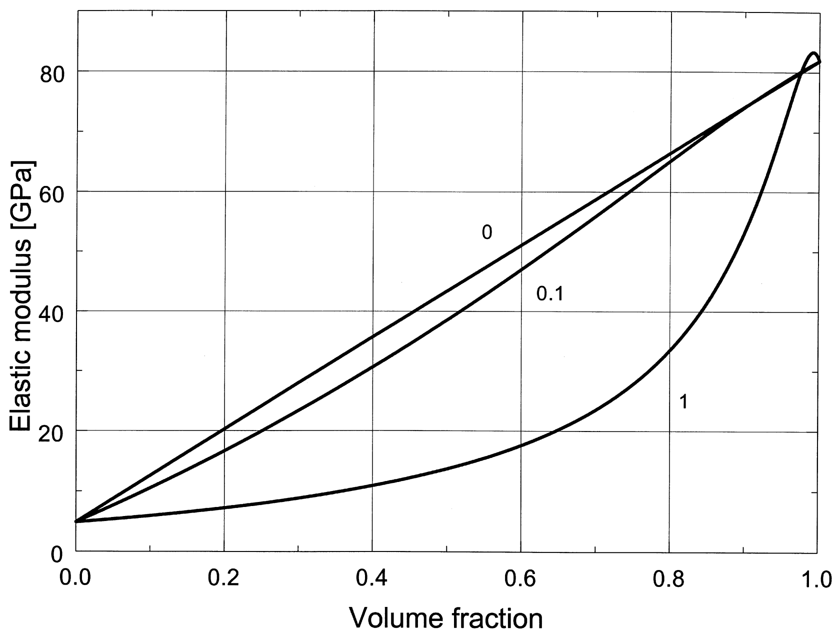

Fig. 2. Effective uniaxial elastic modulus $1/s_{33}^E$ at fixed electric field of a PZT particle reinforced polymer (aligned orientation of inclusions, parameter = particle aspect ratio a/a_3). The material constant s_{33}^E is obtained by normal inversion of the elastic modulus tensor **C** which was introduced in eqn (2.1) for constant electric field.

enced by the piezoelectric coupling acting in the inclusion phase. Figure 5 shows the theoretical predictions for both prolate and oblate particle shape. We found that the local coupling is only important for oblate inclusions (increase by a factor of two for an aspect ratio 100) whereas for spherical or prolate inclusions this influence is insignificant. In Fig. 5 the display range of the volume fraction has been limited to 0.4 since for higher volume fractions the predictions become questionable. This is due to the fact that for a random orientation of inclusions no homogeneous piezoelectric will result if $c \to 1$ since one still has a local property variation by different orientations. Furthermore the main hypothesis of the present EFM becomes senseless when we approach the state of dense packing. So one cannot expect a meaningful limiting result for high volume fraction of inclusions (regardless some special cases as the aligned orientation discussed above). Moreover, for random orientation distribution and high volume fraction, it is known that the Mori–Tanaka method may even predict effective properties outside the general bounds. The same may happen in the present approach though up to now there is no explicit proof for this.

Finally the thermal properties will be considered. Figures 6–8 show some results as functions of the volume fraction of aligned inclusions (note that, according to our definition in (2.1), all pyroelectric constants get a negative numerical value). Generally we may state that the results coincide with the predictions by Dunn (1993) who used the Mori–Tanaka Method.

An interesting behavior occurs for the pyroelectric constant γ_3 at constant stress and constant electric displacement shown in Fig. 7. The theory predicts an increase for decreasing volume fraction up to a maximum followed by a steep decrease to the right zero limit value. For the

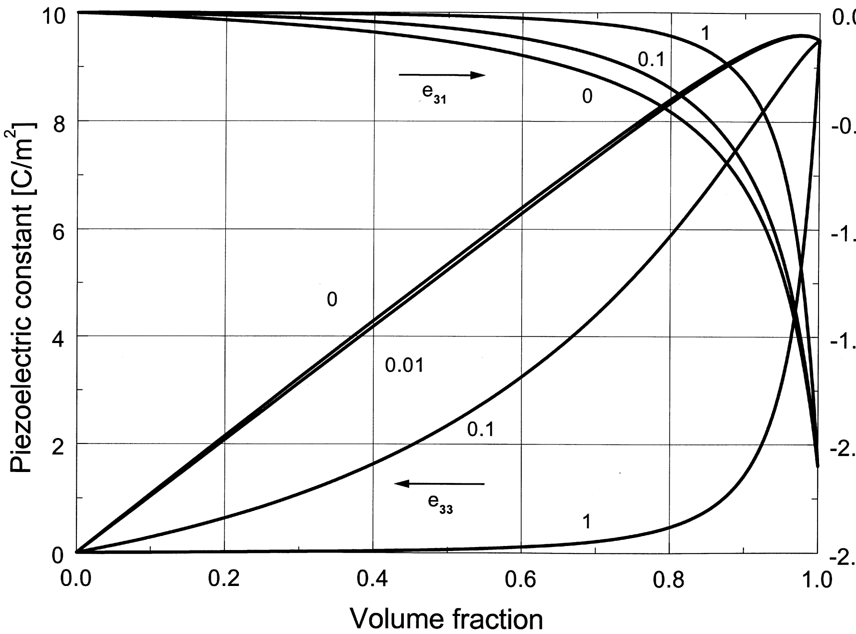

Fig. 3. Effective piezoelectric constants e_{33} and e_{31} of a PZT particle reinforced polymer (random orientation of inclusions, parameter = particle aspect ratio a/a_3).

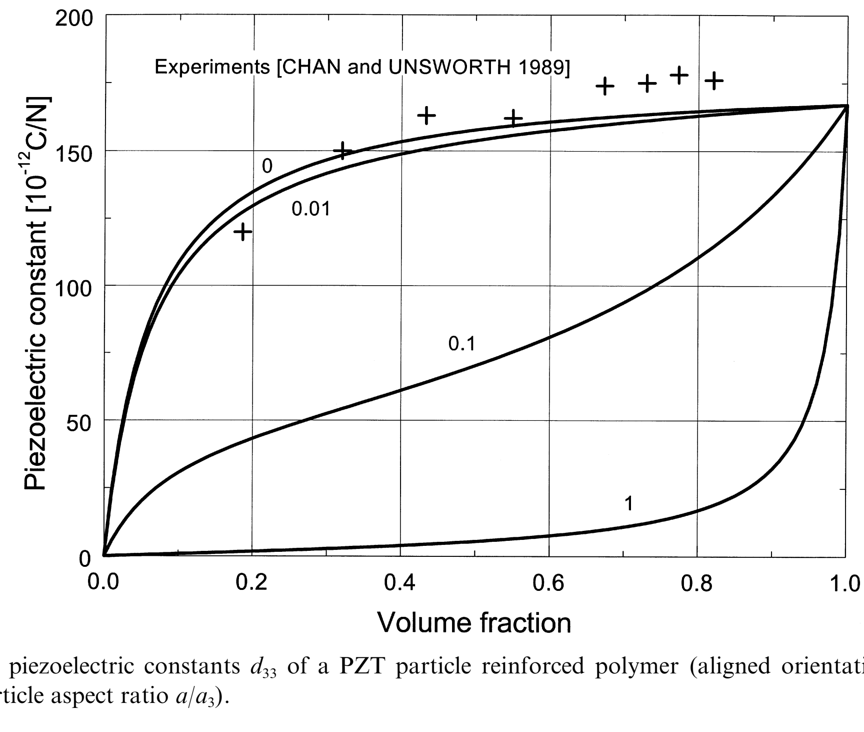

Fig. 4. Effective piezoelectric constants d_{33} of a PZT particle reinforced polymer (aligned orientation of inclusions, parameter = particle aspect ratio a/a_3).

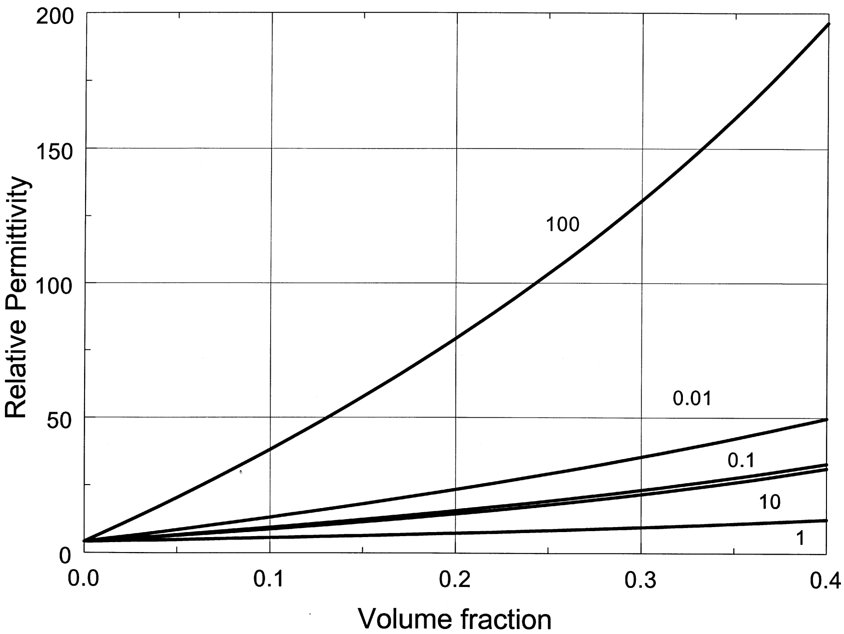

Fig. 5. Effective relative dielectric constant η/ε_0 at fixed strain of a PZT particle reinforced polymer (aligned orientation of inclusions, parameter = particle aspect ratio a/a_3).

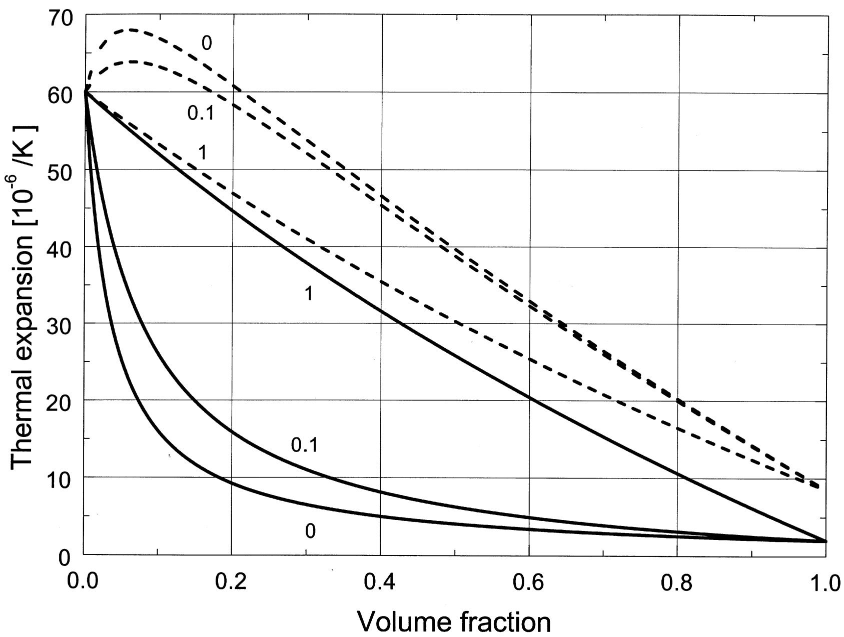

Fig. 6. Effective thermal expansion coefficient ρ_{33} (solid lines) and ρ_{11} (broken lines) at fixed electric displacement of a BT particle reinforced polymer (aligned orientation of inclusions, parameter = particle aspect ratio a/a_3). The material property ρ can be calculated from the other material constants by using eqn (2.4).

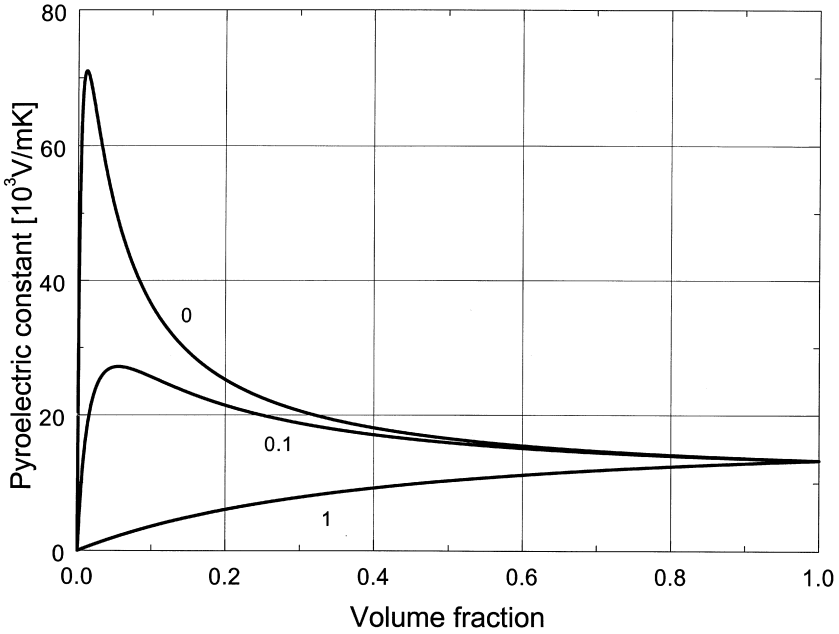

Fig. 7. Effective pyroelectric constant $-\gamma_3$ at fixed stress and electric displacement of a BT particle reinforced polymer (aligned orientation of inclusions, parameter = particle aspect ratio a/a_3). The material property γ can be calculated from the other material constants by using eqn (2.4).

pyroelectric constant at constant electric field and strain, shown in Fig. 8, such a result is not observed. The reason for this unexpected behavior lies in the large thermal expansion coefficient of the polymer matrix: for low volume fractions the fibers have to follow the deformation of the matrix during a temperature change giving rise to high stresses and high electric field in the fibers, which is macroscopically described by the high effective pyroelectric constant.

6. Conclusions

We have shown that the application of the EFM to the coupled thermoelectro-mechanical properties of piezoelectric materials yields reasonable results. The theory is applicable to composites where the anisotropic inclusions in an isotropic matrix may be distributed according to any orientation distribution function. Closed analytical solutions have been presented for complete randomness (uniform orientation distribution) and aligned distribution where no orientation variation occurs.

Because experimental results are rare we have compared our results mainly with other theoretical predictions which are based on the Mori–Tanaka approach. That approach appears from the EFM as a special case when the shape of the correlation hole is identical with the shape of the aligned inclusions. As far as such results are available, we obtained a sufficient coincidence.

Finally we note that the EFM seems also applicable to the determination of the effective

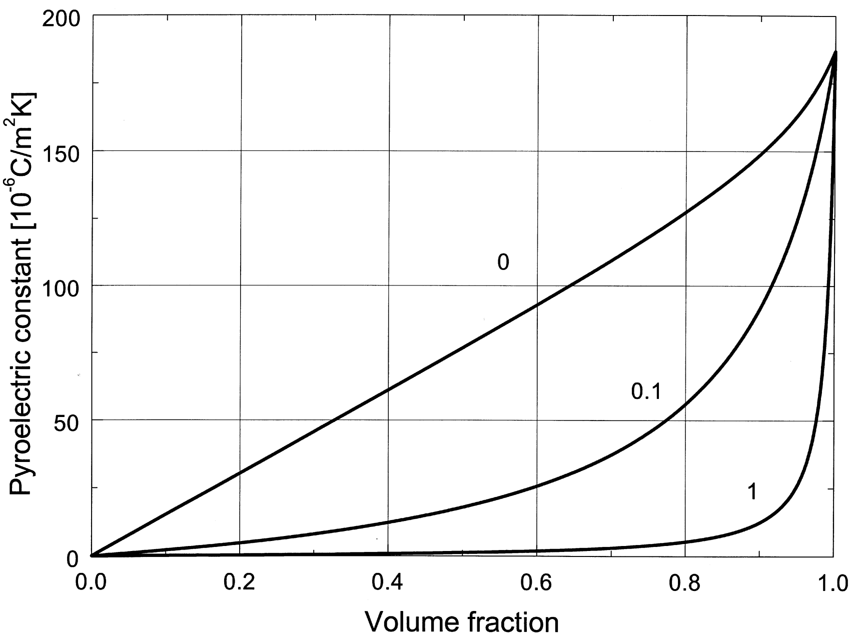

Fig. 8. Effective pyroelectric constant $-\pi_3$ at fixed strain and electric field of a BT particle reinforced polymer (aligned orientation of inclusions, parameter = particle aspect ratio a/a_3).

thermoelectroelastic characteristics of single-phase piezoactive polycrystals. This will be demonstrated in a forthcoming paper.

Appendix: The operator $\boldsymbol{P}^0$ for a spheroidal inclusion in an isotropic matrix

Let us consider an isotropic matrix with Lamé's constants λ_0, μ_0 and the coefficient of dielectric permeability η_0. If the inclusion has a spheroidal shape with semiaxes $a_1 = a_2 = a$, a_3 (the axis of spheroidal symmetry coincides with the x_3-axis of a rectangular system of coordinates), the operator $\boldsymbol{P}^0$ in (2.12) takes the form

$$\boldsymbol{P}^0 = \left\| \begin{matrix} P_{ijkl} & 0 \\ 0 & p_{ik} \end{matrix} \right\| \tag{A.1}$$

where the tensors P_{ijkl} and p_{ik} have hexagonal (or transversely isotropic) symmetry. For the explicit representation of these tensors it is convenient to use the following tensorial bases which are formed by the unit vector m_i of the x_3 axis and the projector $\theta_{ij} = \delta_{ij} - m_i m_j$ on the plane perpendicular to x_3

$$T^1_{ijkl} = \theta_{i(k}\theta_{l)j}, \quad T^2_{ijkl} = \theta_{ij}\theta_{kl}, \quad T^3_{ijkl} = \theta_{ij}m_k m_l, \quad T^4_{ijkl} = m_i m_j \theta_{kl},$$

$$T^5_{ijkl} = \theta_{i)(k}m_{l)}m_{(j}, \quad T^6_{ijkl} = m_i m_j m_k m_l, \quad \theta_{ij} = \delta_{ij} - m_i m_j,$$

$$U^1_{ijk} = \theta_{ij} m_k, \quad U^2_{ijk} = 2m_{(i}\theta_{j)k}, \quad U^3_{ijk} = m_i m_j m_k, \quad t^1_{ij} = m_i m_j, \quad t^2_{ij} = \theta_{ij}. \tag{A.2}$$

The convenience of using the tensorial bases lies in the following properties. The product of the T-basis tensors over two indices forms the closed algebra, the product of the U-basis tensors over one index gives the tensors from T-basis and over two indices—the tensors from t-basis. As for the t-basis, it is orthogonal with respect to contraction by one index, i.e. $t^r_{i\alpha} t^s_{\alpha j} = \delta_{rs} t^r_{ij}$ (no summing on r!).

In these bases we have

$$\mathbf{P} = P_1 \mathbf{T}^2 + P_2(\mathbf{T}^1 - \tfrac{1}{2}\mathbf{T}^2) + P_3(\mathbf{T}^3 + \mathbf{T}^4) + P_5 \mathbf{T}^5 + P_6 \mathbf{T}^6,$$

$$\mathbf{p} = p_1 \mathbf{t}^1 + p_2 \mathbf{t}^2 \tag{A.3}$$

where

$$P_1 = \frac{1}{2\mu_0}[(1-\kappa_0) f_0 + f_1], \quad P_2 = \frac{1}{2\mu_0}[(2-\kappa_0) f_0 + f_1], \quad P_3 = -\frac{f_1}{\mu_0},$$

$$P_5 = \frac{1}{\mu_0}(1 - f_0 - 4f_1), \quad P_6 = \frac{1}{\mu_0}[(1-\kappa_0)(1-2f_0) + 2f_1],$$

$$p_1 = \frac{1}{\eta_0}(1-2f_0), \quad p_2 = \frac{1}{\eta_0} f_0$$

$$f_0 = \frac{1-g}{2(1-\gamma^2)}, \quad f_1 = \frac{\kappa_0}{4(1-\gamma^2)^2}[(2+\gamma^2)g - 3\gamma^2]$$

$$\kappa_0 = \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0}, \quad \gamma = \frac{a}{a_3}. \tag{A.4}$$

In these expressions the function $g(\gamma)$ is determined for oblate spheroids ($\gamma > 1$) by

$$g(\gamma) = \frac{\gamma^2}{\sqrt{\gamma^2 - 1}} \arctan \sqrt{\gamma^2 - 1} \tag{A.5}$$

and for prolate spheroids ($\gamma < 1$) by

$$g(\gamma) = \frac{\gamma^2}{2\sqrt{1-\gamma^2}} \ln \frac{1 + \sqrt{1-\gamma^2}}{1 - \sqrt{1-\gamma^2}}. \tag{A.6}$$

For a spherical inclusion ($\gamma = 1$) the functions f_0 and f_1 go to

$$f_0 = \frac{1}{3}, \quad f_1 = \frac{\kappa_0}{15} \tag{A.7}$$

and the tensors $\mathbf{P}$ and $\mathbf{p}$ become isotropic tensors with the following components

$$P_1 = \frac{5 - 4\kappa_0}{30\mu_0}, \quad P_2 = \frac{1}{2} P_5 = \frac{5 - 2\kappa_0}{15\mu_0}, \quad P_3 = -\frac{\kappa_0}{15\mu_0},$$

$$P_6 = \frac{5-3\kappa_0}{15\mu_0}, \quad p_1 = p_2 = \frac{1}{3\eta_0}. \tag{A.8}$$

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